

M-REGULARITY OF THE FANO SURFACE

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ABSTRACT. In this note we show that the Fano surface in the intermediate Jacobian of a smooth cubic threefold is M -regular in the sense of Pareschi and Popa.

1. INTRODUCTION

Let $X^3 \subset \mathbb{P}^4$ be a smooth cubic threefold, then its intermediate Jacobian

$$J(X) := H^{2,1}(X, \mathbb{C})^* / H_3(X, \mathbb{Z})$$

is a principally polarised abelian variety $(J(X), \Theta)$ of dimension five that is not a Jacobian of a curve [4, Thm.0.12]. The Fano scheme F parametrising lines contained in X is a smooth surface, and the Abel-Jacobi map $F \rightarrow J(X)$ is an embedding that induces an isomorphism $\text{Alb}(F) \simeq J(X)$ [4, Thm.0.6,0.9]. Furthermore the cohomology class of $F \subset J(X)$ is minimal, that is

$$[F] = \frac{\Theta^3}{3!}.$$

There is only one other known family of examples of principally polarised abelian varieties (A, Θ) of dimension n such that for $1 \leq d \leq n-2$, a minimal cohomology class $\frac{\Theta^{n-d}}{(n-d)!}$ can be represented by an effective cycle of dimension d : the Jacobians of curves $J(C)$ where the subvarieties $W_d(C) \subset J(C)$ have minimal cohomology class. O. Debarre has shown that on a Jacobian these are the only subvarieties having minimal class [5, Thm.5.1], furthermore by a theorem of Z. Ran [11, Thm.5], the only principally polarised abelian fourfolds with a subvariety of minimal class are (products of) Jacobians of curves. In higher dimension few things are known about subvarieties having minimal class.

In [9], G. Pareschi and M. Popa introduce a new approach to the characterisation of these subvarieties: they consider the (probably more tractable) cohomological properties of the twisted structure sheaf of the subvariety. More precisely we have the following conjecture.

1.1. Conjecture. [5],[9] *Let (A, Θ) be an irreducible principally polarised abelian variety of dimension n , and let Y be a nondegenerate subvariety (cf. [11, p.464]) of A of dimension $d \leq n-2$. The following statements are equivalent.*

- 1.) *The variety Y has minimal cohomology class, i.e. $[Y] = \frac{\Theta^{n-d}}{(n-d)!}$.*
- 2.) *The twisted structure sheaf $\mathcal{O}_Y(\Theta)$ is M -regular (cf. definition 1.4 below), and $h^0(Y, \mathcal{O}_Y(\Theta) \otimes P_\xi) = 1$ for $P_\xi \in \text{Pic}^0(A)$ general.*

- 3.) Either (A, Θ) is the Jacobian of a curve of genus n and Y is a translate of $W_d(C)$ or $-W_d(C)$, or $n = 5$, $d = 2$ and (A, Θ) is the intermediate Jacobian of a smooth cubic threefold and Y is a translate of F or $-F$.

The implication $2) \Rightarrow 1)$ is the object of [9, Thm.B]. The implication $3) \Rightarrow 2)$ has been shown for Jacobians of curves in [8, Prop.4.4]. We complete the proof of this implication by treating the case of the intermediate Jacobian.

1.2. Theorem. *Let $X^3 \subset \mathbb{P}^4$ be a smooth cubic threefold, and let $(J(X), \Theta)$ be its intermediate Jacobian. Let $F \subset J(X)$ be an Abel-Jacobi embedded copy of the Fano variety of lines in X . Then $\mathcal{O}_F(\Theta)$ is M -regular and $h^0(F, \mathcal{O}_F(\Theta) \otimes P_\xi) = 1$ for $P_\xi \in \text{Pic}^0 J(X)$ general.*

Since the properties considered are invariant under isomorphisms, the theorem implies the same statement for $-F$.

The study of the remaining open implications of conjecture 1.1 is a much harder task than the proof of theorem 1.2. In an upcoming paper we will start to investigate this problem under the additional hypothesis that (A, Θ) is the intermediate Jacobian of a generic smooth cubic threefold. In this case we can show the following statement.

1.3. Theorem. [6] *Let $X^3 \subset \mathbb{P}^4$ be a general smooth cubic threefold. Let $(J(X), \Theta)$ be its intermediate Jacobian, and let $F \subset J(X)$ be an Abel-Jacobi embedded copy of the Fano variety of lines in X . Let $S \subset J(X)$ be a surface that has minimal cohomology class, i.e. $[S] = \frac{\Theta^3}{3!}$. Then S is a translate of F or $-F$.*

Notation and basic facts.

We work over an algebraically closed field of characteristic different from 2. We will denote by $D \equiv D'$ the linear equivalence of divisors, and by $D \equiv_{\text{num}} D'$ the numerical equivalence.

For (A, Θ) a principally polarised abelian variety (ppav), we identify A with $\hat{A} = \text{Pic}^0(A)$ via the morphism induced by Θ . If $\xi \in A$ is a point, we denote by P_ξ the corresponding point in $\hat{A} = \text{Pic}^0(A)$ which we consider as a numerically trivial line bundle on A .

1.4. Definition. [10] *Let (A, Θ) be a ppav of dimension n , and let \mathcal{F} be a coherent sheaf on A . For all $n \geq i > 0$, we denote by*

$$V_{\mathcal{F}}^i := \{\xi \in A \mid h^i(A, \mathcal{F} \otimes P_\xi) > 0\}$$

the i -th cohomological support locus of \mathcal{F} . We say that \mathcal{F} is M -regular if

$$\text{codim} V_{\mathcal{F}}^i > i$$

for all $i \in \{1, \dots, n\}$.

If $l \subset X$ is a line, we will denote by $[l]$ the corresponding point of the Fano surface F and by $D_l \subset F$ the incidence curve of l , that is, D_l parametrises lines in X that meet l . Furthermore we have by [4, §10], [12, §6] and Riemann-Roch that

$$(1.5) \quad \mathcal{O}_F(\Theta) \equiv_{\text{num}} 2D_l,$$

$$(1.6) \quad K_F \equiv_{\text{num}} 3D_l,$$

$$(1.7) \quad D_l \cdot D_l = 5,$$

$$(1.8) \quad \chi(F, \mathcal{O}_F(\Theta)) = 1.$$

2. PRYM CONSTRUCTION OF THE FANO SURFACE

We recall the construction of the Fano surface as a special subvariety of a Prym variety [3, 2]: let $\tilde{C} := D_{l_0} \subset F$ be the incidence curve of a general line $l_0 \subset X$. Let X' be the blow-up of X in l_0 . Then the projection from l_0 induces a conic bundle structure $X' \rightarrow \mathbb{P}^2$ with branch locus $C \subset \mathbb{P}^2$ a smooth quintic. This conic bundle induces a natural connected étale covering of degree two $\pi : \tilde{C} \rightarrow C$ (cf. [1, Ch.I] for details), and we denote by $\sigma : \tilde{C} \rightarrow \tilde{C}$ the involution induced by π .

The kernel of the norm morphism $\text{Nm} : J\tilde{C} \rightarrow JC$ has two connected components which we will denote by P and P_1 . The zero component P is called the Prym variety associated to π , and it is isomorphic as a ppav to $J(X)$ [1, Thm.2.1].

Let $H \subset C$ be an effective divisor given by a hyperplane section in \mathbb{P}^2 . Then H has degree five and $h^0(C, \mathcal{O}_C(H)) = 3$, so the complete linear system g_5^2 corresponds to a $\mathbb{P}^2 \subset C^{(5)}$. We choose a divisor $\tilde{H} \in \tilde{C}^{(5)}$ such that $\pi^{(5)}([\tilde{H}]) = [H]$, where $\pi^{(5)} : \tilde{C}^{(5)} \rightarrow C^{(5)}$ is the morphism induced by π on the symmetric products. Let $\phi_H : C^{(5)} \rightarrow JC$ and $\phi_{\tilde{H}} : \tilde{C}^{(5)} \rightarrow J\tilde{C}$ be the Abel-Jacobi maps given by H and \tilde{H} . We have a commutative diagram

$$\begin{array}{ccc} \tilde{C}^{(5)} & \xrightarrow{\phi_{\tilde{H}}} & J\tilde{C} \\ \pi^{(5)} \downarrow & & \downarrow \text{Nm} \\ C^{(5)} & \xrightarrow{\phi_H} & JC \end{array}$$

The fibre of $\phi_{\tilde{H}}(\tilde{C}^{(5)}) \rightarrow \phi_H(C^{(5)})$ over the point 0 (and thus the intersection of $\phi_{\tilde{H}}(\tilde{C}^{(5)})$ with $\ker \text{Nm}$) has two connected components $F_0 \subset P$ and $F_1 \subset P_1$. If we identify P and P_1 via $\tilde{H} - \sigma(\tilde{H})$, we obtain an identification $F_1 = -F_0$ [3, p.360]. The (non-canonical) isomorphism of ppavs $P \simeq J(X)$ transforms F_0 into a translate of the Fano surface F [3, Thm.4].

From now on we will identify P (resp. F_0) and $J(X)$ (resp. some Abel-Jacobi emdded copy of the Fano surface F).

We will now prove two technical lemmata on certain linear systems on \tilde{C} . The first is merely a reformulation of [2, §2,ii)].

2.9. Lemma. *The line bundle $\mathcal{O}_{\tilde{C}}(\tilde{C})$ is a base-point free pencil of degree five such that any divisor $D \in |\mathcal{O}_{\tilde{C}}(\tilde{C})|$ satisfies $\pi_* D \equiv H$.*

Proof. We define a morphism $\mu : \tilde{C} = D_{l_0} \rightarrow l_0 \simeq \mathbb{P}^1$ by sending $[l] \in \tilde{C}$ to $l \cap l_0$. Since l_0 is general and through a general point of l_0 there are five lines distinct from l_0 , the morphism μ has degree 5. If $[l] \in F$, then $D_l \cdot D_{l_0} = 5$ by formula (1.7), so for $[l] \neq [l_0]$ the divisor $D_{l_0} \cap D_l \in |\mathcal{O}_{\tilde{C}}(D_l)|$ is effective. Furthermore $\pi_* D_l \equiv H$, since $\pi_* D_l$ is the intersection of $C \subset \mathbb{P}^2$ with the image of l under the projection $X' \rightarrow \mathbb{P}^2$. By specialisation the linear system $|\mathcal{O}_{\tilde{C}}(\tilde{C})|$ is not empty and a general divisor D in it corresponds to the five lines distinct from l_0 passing through a general point of l_0 . Hence $\mathcal{O}_{\tilde{C}}(\tilde{C}) \simeq \mu^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $\pi_* D \equiv H$. \square

2.10. Lemma. *The sets*

$$V'_0 := \{\xi \in P \mid h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\tilde{C}) \otimes P_\xi) > 0\}$$

$$V'_1 := \{\xi \in P \mid h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi) > 1\}$$

are contained in translates of $F \cup (-F)$.

Proof.

1) Let $D \in |\mathcal{O}_{\tilde{C}}(\tilde{C}) \otimes P_\xi|$ be an effective divisor. Then $\pi_*\tilde{C} \equiv \pi_*D \equiv H$. It follows that $D \in (\phi_{\tilde{H}}(\tilde{C}^{(5)}) \cap \ker \text{Nm})$, so D is in F or $-F$.

2) We follow the argument in [2, §3]. By [2, §2,iv)] we have $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\tilde{C} + \sigma(\tilde{C}))) = 4$, so $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}))$ is odd. It follows from the deformation invariance of the parity [7, p.186f] that

$$V'_1 = \{\xi \in P \mid h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi) \geq 3\}.$$

Fix $\xi \in P$ such that $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi) \geq 3$ and $D \in |\mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi|$. Let s and t be two sections of $\mathcal{O}_{\tilde{C}}(\tilde{C})$ such that the associated divisors have disjoint supports, then we have an exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{C}}(D - \tilde{C}) \xrightarrow{(t, -s)} \mathcal{O}_{\tilde{C}}(D)^{\oplus 2} \xrightarrow{(s, t)} \mathcal{O}_{\tilde{C}}(D + \tilde{C}) \rightarrow 0.$$

This implies

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D - \tilde{C})) + h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D + \tilde{C})) \geq 2h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D)) = 6,$$

furthermore by Riemann-Roch $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D + \tilde{C})) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(K_{\tilde{C}} - D - \tilde{C})) + 5$. Now $K_{\tilde{C}} - D \equiv \sigma(D)$ and $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\sigma(D) - \tilde{C})) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D - \sigma(\tilde{C})))$ imply

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D - \tilde{C})) + h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D - \sigma(\tilde{C}))) \geq 1.$$

Hence $D \equiv \tilde{C} + D'$ or $D \equiv \sigma(\tilde{C}) + D'$ where D' is an effective divisor such that $\pi_*D' \equiv H$. We see as in the first part of the proof that the effective divisors D' such that $\pi_*D' \equiv H$ are parametrised by a set that is contained in a translate of $F \cup (-F)$. \square

3. PROOF OF THEOREM 1.2.

Since $\mathcal{O}_F(\Theta) \equiv_{\text{num}} \mathcal{O}_F(2\tilde{C})$ by formula (1.5), it is equivalent to verify the stated properties for the sheaf $\mathcal{O}_F(2\tilde{C})$.

Step 1. The second cohomological support locus is contained in a translate of $F \cup (-F)$. By formula (1.6), we have $K_F \equiv \mathcal{O}_F(3\tilde{C}) \otimes P_{\xi_0}$ for some $\xi_0 \in P$. Hence by Serre duality $h^2(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) = h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi^* \otimes P_{\xi_0})$, so it is equivalent to consider the non-vanishing locus

$$V_0 := \{\xi \in P \mid h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi) > 0\}.$$

If $l \in F$ is a line on X , the corresponding incidence curve $D_l \subset F$ is an effective divisor numerically equivalent to \tilde{C} , so it is clear that $\pm F$ is (up to translation) a subset of V_0 . In order to show that we have an equality, consider the exact sequence

$$0 \rightarrow \mathcal{O}_F \otimes P_\xi \rightarrow \mathcal{O}_F(\tilde{C}) \otimes P_\xi \rightarrow \mathcal{O}_{\tilde{C}}(\tilde{C}) \otimes P_\xi \rightarrow 0.$$

Clearly $h^0(F, \mathcal{O}_F \otimes P_\xi) = 0$ for $\xi \neq 0$, so $h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi) \leq h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\tilde{C}) \otimes P_\xi)$ for $\xi \neq 0$. Since a divisor $D \in |\mathcal{O}_{\tilde{C}}(\tilde{C})|$ satisfies $\pi_*D \equiv H$, we conclude with Lemma 2.10.

Step 2. The first cohomological support locus is contained in a union of translate of $F \cup (-F)$. Since $\chi(F, \mathcal{O}_F(2\tilde{C})) = \chi(F, \mathcal{O}_F(\Theta)) = 1$ (formula (1.8)), we have

$$h^1(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) = h^0(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) + h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi^* \otimes P_{\xi_0}) - 1.$$

Since

$$h^0(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) = h^0(F, \mathcal{O}_F(\Theta) \otimes P_\xi) \geq 1$$

for all $\xi \in P$, the first cohomological support locus is contained in the locus where $h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi^* \otimes P_{\xi_0}) > 0$ or $h^0(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) > 1$. By step 1 the statement follows if we show the following claim: the set

$$V_1 := \{\xi \in P \mid h^0(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) > 1\}$$

is contained in a union of translates of $F \cup (-F)$.

Step 3. Proof of the claim and conclusion. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_F(\tilde{C}) \otimes P_\xi \rightarrow \mathcal{O}_F(2\tilde{C}) \otimes P_\xi \rightarrow \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi \rightarrow 0.$$

By the first step we know that $h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi) = 0$ for ξ in the complement of a translate of $F \cup (-F)$, so

$$h^0(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) \leq h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi)$$

for ξ in the complement of a translate of $F \cup (-F)$. The claim is then immediate from Lemma 2.10. By the same lemma $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi) = 1$ for $\xi \in P$ general, so $h^0(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) = h^0(F, \mathcal{O}_F(\Theta) \otimes P_\xi) = 1$ for $\xi \in P$ general. \square

Remark. It is possible to strengthen *a posteriori* the statements in the proof: since Theorem 1.2 holds, we can use the Fourier-Mukai techniques from [9] to see that the cohomological support loci are supported exactly on the *theta-dual* of F (ibid, Definition 4.2), which in our case is just F .

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REFERENCES

- [1] A. Beauville. Variétés de Prym et jacobiniennes intermédiaires. *Ann. Sci. École Norm. Sup. (4)*, 10(3):309–391, 1977.
- [2] A. Beauville. Les singularités du diviseur Θ de la jacobienne intermédiaire de l’hypersurface cubique dans \mathbf{P}^4 . In *Lect. Notes Math.* 947., pages 190–208. Springer, Berlin, 1982.
- [3] A. Beauville. Sous-variétés spéciales des variétés de Prym. *Comp. Math.*, 45(3):357–383, 1982.
- [4] C. H. Clemens and P. A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.
- [5] O. Debarre. Minimal cohomology classes and Jacobians. *J. Alg. Geom.*, 4(2):321–335, 1995.
- [6] A. Höring. Paper in preparation. *Soon on this server*, 2007.
- [7] D. Mumford. Theta characteristics of an algebraic curve. *Ann. Sci. École Norm. Sup. (4)*, 4:181–192, 1971.
- [8] G. Pareschi and M. Popa. Regularity on abelian varieties. I. *J. Amer. Math. Soc.*, 16(2):285–302, 2003.
- [9] G. Pareschi and M. Popa. Generic vanishing and minimal cohomology classes on abelian varieties. *arXiv:math.AG/0610166*, 2006.
- [10] G. Pareschi and M. Popa. GV-sheaves, Fourier-Mukai transform, and Generic Vanishing. *arXiv:math.AG/0608127*, 2006.
- [11] Z. Ran. On subvarieties of abelian varieties. *Inventiones Math.*, 62:459–479, 1981.
- [12] G. E. Welters. *Abel-Jacobi isogenies for certain types of Fano threefolds*, volume 141 of *Mathematical Centre Tracts*. Mathematisch Centrum, Amsterdam, 1981.

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